Exact Results for Average Cluster Numbers in Bond Percolation on Lattice Strips

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We present exact calculations of the average number of connected clusters per site, $\langle k \rangle$, as a function of bond occupation probability p, for the bond percolation problem on infinite-length strips of finite width L_y, of the square, triangular, honeycomb, and kagomé lattices Λ with various boundary conditions. These are used to study the approach of $\langle k \rangle$, for a given p and Λ, to its value on the two-dimensional lattice as the strip width increases. We investigate the singularities of $\langle k \rangle$ in the complex p plane and their influence on the radii of convergence of the Taylor series expansions of $\langle k \rangle$ about $p = 0$ and $p = 1$.

I. INTRODUCTION

The study of percolation gives insight into a number of important phenomena such as the passage of fluids through porous media and the effect of lattice defects and disorder on critical phenomena. Here we consider bond percolation. Let $G = G(V, E)$ be a connected graph defined by a set V of vertices (sites) and a set E of edges (bonds) connecting pairs of vertices. Multiple bonds connecting the same pair of vertices are allowed, although most of the graphs that we study have only simple bonds. We denote the number of vertices and bonds as $n = n(G) = |V|$ and $e(G) = |E|$. One now envisions generating a large ensemble of corresponding graphs in which each bond is randomly present with probability $p \in [0, 1]$. In the usual statistical mechanics context, one is interested in a d-dimensional thermodynamic limit of a regular lattice graph Λ in which the bonds are present with probability p. Consider the probability $P(\Lambda, p)$ that a given site belongs to an infinite cluster. For a given lattice Λ, as p decreases from 1, $P(\Lambda, p)$ decreases monotonically until, at a critical value, $p_{c,\Lambda}$, it vanishes and remains identically zero for $0 \le p \le p_{c,\Lambda}$. This has an important effect on physical phenomena that take place on such a bond-diluted lattice. For example, consider a spin-model (with finite-range spin-spin interactions) on Λ and assume that for the undiluted lattice the model is above its lower critical dimensionality so that there is a phase transition at a finite temperature T_c. Then as p decreases below unity, $T_c(p)$ decreases below $T_c(1)$, and as p decreases to $p_{c,\Lambda}$, $T_c(p)$ vanishes nonanalytically and remains zero for $0 \le p \le p_{c,\Lambda}$. Other quantities also behave nonanalytically at $p_{c,\Lambda}$, such as the mean size

$S(\Lambda, p)$ of the percolation cluster, which diverges as p increases through $p_{c,\Lambda}$. Some previous literature on bond percolation relevant to our present work includes Refs. [1]-[26].

An interesting quantity in this context is the number of connected components (clusters), including single sites, for a given lattice Λ, divided by the number of sites on the lattice and averaged over all of the graphs in the above ensemble. We denote this mean cluster number per site as $\langle k \rangle_\Lambda$. Explicit exact values for $\langle k \rangle_\Lambda$ at the respective values $p = p_{c,\Lambda}$, denoted $\langle k \rangle_{c,\Lambda}$, have been calculated in Ref. [22] for the square, triangular, and honeycomb lattices (where they were denoted $n_c^{B-\Lambda}$). However, to our knowledge, $\langle k \rangle$ has never been calculated exactly as a function of p for the full range $p \in [0, 1]$ for any lattice except a one-dimensional or Bethe lattice [2, 11, 17].

In this paper we present exact calculations of this average cluster number per site, $\langle k \rangle$, as a function of p, for a variety of infinite-length, finite-width strips of regular lattices. We consider strips of the square, triangular, honeycomb, and kagomé lattices. These are of interest since, at least for modest strip widths, one can obtain explicit analytic expressions for $\langle k \rangle$ and can exactly determine, e.g., singularities that these expressions have in the complex p plane and their influence on series expansions. When referring to a specific lattice strip Λ_s (including transverse boundary conditions), we shall denote the average cluster number per site as $\langle k \rangle_{\Lambda_s}$. Our results interpolate between the known exact solutions for the one-dimensional lattice (line) and the case of two dimensions, and complement numerical simulations and series expansions. Early studies of percolation on infinite-length, finite-width strips include Refs. [14, 15], which focused on universal properties such as critical exponents. While the average cluster

number is obviously nonuniversal, depending as it does on the specific type of lattice, it still give useful information about the phenomenon of percolation, as we shall show.

We take the longitudinal and transverse directions to be x and y and denote the size of the lattice strips in these directions as L_x and L_y and the respective boundary conditions as BC_x and BC_y. We focus on the limit of infinite length, $L_x \to \infty$, for which the results are independent of the longitudinal boundary conditions. For an infinite-length strip of a lattice Λ, as the width $L_y \to \infty$, one expects $\langle k \rangle$ to approach a limiting function of p which is independent of the transverse boundary conditions and is equal to $\langle k \rangle$ for the corresponding infinite two-dimensional lattice Λ. In particular, for a given infinite-length, finite-width strip of the lattice Λ, it is of interest to evaluate our exact expressions for $\langle k \rangle$ at $p = p_{c,\Lambda}$ and study how the resultant value approaches the critical value $\langle k \rangle_{c,\Lambda}$ for the corresponding infinite two-dimensional lattice.

II. CALCULATIONAL METHOD

For a given graph $G = (V, E)$ we calculate $\langle k \rangle$ by making use of the fact that it can be expressed as a certain derivative of the (reduced) free energy of the Potts model. We first recall this expression. Define a spanning subgraph $G' = (V, E')$ as a subgraph of G with the same vertex set V and a subset of the edge (bond) set, $E' \subseteq E$. From the definition of the average number of clusters per site, we have

$$\langle k \rangle = \frac{(1/n) \sum_{G'} k(G') p^{e(G')} (1-p)^{e(G) - e(G')}}{\sum_{G'} p^{e(G')} (1-p)^{e(G) - e(G')}}$$

$$= \frac{(1/n) \sum_{G'} k(G') [p/(1-p)]^{e(G')}}{\sum_{G'} [p/(1-p)]^{e(G')}} \ . \qquad (2.1)$$

This follows because each G' contains $k(G')$ connected components, and appears in the numerator of the expression in the first line with weight given by $p^{e(G')} (1 - p)^{e(G) - e(G')}$, since the probability that all of the bonds in G' are present is $p^{e(G')}$ and the probability that all of the other $e(G) - e(G')$ bonds in G are absent is $(1-p)^{e(G) - e(G')}$. This sum in the numerator over the set of spanning subgraphs G' is normalized by the indicated denominator and by an overall factor of $1/n$ to obtain the average number of connected components (clusters) per site. As noted, we shall focus on the limit in which the strip length $L_x \to \infty$ and hence $n \to \infty$.

The cluster representation for the partition function of

the q-state Potts model is [7, 16]

$$Z(G, q, v) = \sum_{G' \subseteq G} q^{k(G')} v^{e(G')} \ . \qquad (2.2)$$

In the Potts spin model, the spin σ_i at each vertex i can take on q different values $\sigma_i = 1, 2, .., q$ and

$$v \equiv e^K - 1 \ , \quad K = \beta J \ , \qquad (2.3)$$

where $\beta = 1/(k_B T)$ with T the temperature and J the spin-spin exchange constant. The formula (2.2) allows one to generalize q from the positive integers to a non-negative real variable while retaining a Gibbs measure for $Z(G, q, v)$.

On a finite graph G one defines the (reduced) free energy per site of the Potts model as

$$f(G, q, v) = \ln[Z(G, q, v)^{1/n}] \qquad (2.4)$$

and, in the limit $n \to \infty$,

$$f(\{G\}, q, v) = \lim_{n \to \infty} f(G, q, v) \qquad (2.5)$$

where we use the symbol $\{G\}$ to denote the formal limit $\lim_{n \to \infty} G$ for a given family of graphs. (The actual free energy per site is $F = -k_B T f$; henceforth we shall simply refer to f as the free energy.) If one sets

$$v = v_p \equiv \frac{p}{1-p} \ , \qquad (2.6)$$

differentiates $f(G, q, v_p)$ with respect to q, and sets $q = 1$, one obtains precisely the expression for $\langle k \rangle_n$, as given in eq. (2.1), i.e.,

$$\langle k \rangle_n = \frac{\partial f(G, q, v_p)}{\partial q} \bigg|_{q=1} \ . \qquad (2.7)$$

In particular, in the limit $n \to \infty$,

$$\langle k \rangle = \frac{\partial f(\{G\}, q, v_p)}{\partial q} \bigg|_{q=1} \ . \qquad (2.8)$$

Our method for calculating $\langle k \rangle$ is to use eq. (2.8) in conjunction with exact results that we have computed for the free energy of the Potts model on infinite-length, finite-width strips of various lattices [27]-[37]. One can also study quantities in the percolation problem in an analytic manner without making use of eq. (2.8) (e.g., [14]), but the method we use suffices for our purposes. One of the interesting properties of the formula (2.8) is that it relates a geometric property of (the $n \to \infty$ limit of) a bond-diluted graph with the derivative of the Potts model, evaluated at a certain temperature as $q \to 1$, on a graph with no bond dilution. Specifically, the mapping (2.6), in conjunction with eq. (2.8), formally associates

the interval $0 \le p \le 1$ with the interval $0 \le v \le \infty$, which is the physical range of values of the temperature variable v for the ferromagnetic Potts model. In passing, we recall that the connection of percolation to the $q = 1$ Potts model implies that the percolation transition on 2D lattices is second-order, in the same universality class as the latter model, with the exactly known thermal and magnetic critical exponents $y_t = 3/4$ and $y_h = 91/48$ and thus $\alpha = \alpha' = -2/3$, $\beta = 5/36$, $\gamma = \gamma' = 43/18$, $\delta = 91/5$, etc., the values of which were later illuminated further by the application of conformal algebra.

III.   SOME BASIC PROPERTIES OF $\langle k \rangle$

As background for our exact results, it will be useful to note some basic properties of $\langle k \rangle$. These can all be understood using elementary methods. First, in the limit as $p \to 0$, the only nonzero contribution to the sum in the numerator, and to the sum in the denominator, of eq. (2.1) arises from the spanning subgraph G' with no edges, $e(G') = \emptyset$, which has n connected components, so $\lim_{p\to 0} \langle k \rangle_n = \lim_{p\to 0} \langle k \rangle = 1$. Second, in the limit as $p \to 1$, the only nonzero contribution to the sum in the numerator, and to the sum in the denominator, of eq. (2.1) arises from the term in which the spanning subgraph G' is G itself, with $k(G) = 1$, so $\lim_{p\to 1} \langle k \rangle_n = 1/n$ and hence $\lim_{p\to 1} \langle k \rangle = 0$. Third, for a given $\{G\}$, the average per-site cluster number $\langle k \rangle$ on $\{G\}$ is a monotonically decreasing function of p for $p \in [0, 1]$. To see this, we start with $\langle k \rangle_n$ on a finite G and observe that increasing p in the interval $[0, 1]$ decreases the number of components $k(G)$. Dividing by n and taking the limit $n \to \infty$ yields the corresponding monotonicity result for $\langle k \rangle$ on $\{G\}$.

Further, for the infinite-length, finite-width strips considered here, one can show that $\langle k \rangle$ is a (real) analytic function of p for $p \in [0, 1)$ as follows. We use the general relation eq. (2.8) and observe that the mapping (2.6) formally associates the value $p = 0$ in the bond percolation problem with the value $v = 0$, i.e., infinite temperature, in the Potts model free energy. Since the free energy of a spin model is always analytic near $\beta = 0$, i.e., $v = 0$, it follows that for any $\{G\}$, $\langle k \rangle$ is analytic in the neighborhood of $p = 0$. As p increases from 0 to 1, the value of v corresponding to it via the mapping (2.6) increases from $v = 0$ to $v = \infty$ (i.e., $T = 0$). Since infinite-length, finite-width lattice strip graphs are quasi-one-dimensional, we can use the property that the free energy of a quasi-one-dimensional spin system with finite-range spin-spin interactions, such as the Potts model, is analytic for all finite temperatures. It follows that $\langle k \rangle$ is analytic for $0 \le p < 1$. In general, for quasi-1D systems, the critical percolation probability is $p_c = 1$, so

that many quantities in 1D percolation are nonanalytic at $p = 1$; examples include the average cluster size function $S(p) = (1 + p)/(1 - p)$, which diverges as $p \to 1^-$, and the percolation probability $P(p)$, which is zero for $0 \le p < 1$ and jumps discontinuously to 1 at $p = 1$ [2, 11, 17]. This is analogous to the fact that the critical temperature for ferromagnetic spin models with short-range spin-spin interactions in 1D is $T_c = 0$ ($v = \infty$). However, for all of the cases that we have calculated, the mean cluster number $\langle k \rangle$, is analytic at $p = 1$ as well as for $p \in [0, 1)$. This is somewhat similar to the situation with the internal energy of a ferromagnetic 1D spin system with short-range spin-spin interactions, which is analytic at $T = 0$ even though the susceptibility and correlation lengthy diverge as $T \to 0$ and the magnetization jumps from 0 to 1 at $T = 0$. This analyticity of $\langle k \rangle$ for $p \in [0, 1]$ for infinite-length, finite-width lattice strips is, of course, different from the situation on lattices Λ of dimensionality $d \ge 2$, where $\langle k \rangle$ generically has a (finite) nonanalyticity at $p = p_{c,\Lambda}$. The fact that $\langle k \rangle$ for the infinite-length, finite-width lattice strips is analytic for $p \in [0, 1]$ does not prevent this quantity from having singularities at unphysical values of p, including real values outside the interval $0 \le p \le 1$ and complex values. These will be discussed below.

The property that $\langle k \rangle$ (calculated in the limit as $L_x \to \infty$) is independent of the longitudinal boundary conditions imposed on the lattice strip follows from the same property for the Potts model free energy. As noted above, we further expect that, for a given transverse boundary condition and for a given $p \in [0, 1]$, the value of $\langle k \rangle$ for an infinite-length strip of the lattice Λ approaches the corresponding value of $\langle k \rangle$ for the (infinite) two-dimensional lattice Λ as $L_y \to \infty$. This means that for moderate finite widths L_y, the resultant $\langle k \rangle$ can serve as exactly calculated approximation to the value of $\langle k \rangle$ for the corresponding two-dimensional lattice. Of course, this approximation refers to the value only, not the full analytic structure; as we have shown, for any finite L_y, regardless of how large, $\langle k \rangle$ does not contain the (finite) nonanalyticity at $p_{c,\Lambda}$ that is present in $\langle k \rangle$ for the infinite two-dimensional lattice.

IV.   RELEVANT RESULTS FOR BOND PERCOLATION ON TWO-DIMENSIONAL LATTICES

A.   Values of $p_{c,\Lambda}$

Here we briefly recall some results that we shall use for bond percolation on two-dimensional lattices. For a planar lattice G, the (planar) dual, G^* is the lattice

whose vertices and faces are given, respectively, by the faces and vertices of G and whose edges link the faces of G. For the thermodynamic limit of a regular two-dimensional lattice graph G, the critical probability p_c for bond percolation satisfies [3] $p_{c,\{G\}} + p_{c,\{G^*\}} = 1$. In particular, it follows that, since the square (sq) lattice is self-dual, $p_{c,sq} = 1/2$. The values of p_c are also known exactly for the triangular (tri) and honeycomb (hc) lattices [3]: $p_{c,tri} = 2\sin(\pi/18) = 0.347296..$ and $p_{c,hc} = 1 - p_{c,tri} = 0.6527036...$

It is useful to recall that for a general lattice Λ, $p_{c,\Lambda}$ can be obtained from a knowledge of the critical temperature of the $q \to 1$ Potts ferromagnet. The relation (2.6) is equivalently written as $p = v/(1+v)$. Hence, the critical percolation probability is determined as $p_{c,\Lambda} = v_{c,\Lambda}/(1+v_{c,\Lambda})$, where $v_{c,\Lambda}$ is the value of v corresponding to the phase transition temperature of the Potts ferromagnet on Λ in the limit $q \to 1$. The value of $v_{c,\Lambda}$ is known exactly for the square, triangular and honeycomb lattices [16]. Thus, v_c for the square lattice is given by the self-duality relation $v = \sqrt{q}$; setting $q = 1$ yields $v_{c,sq} = 1$ and substituting this into eq. (2.6) yields $p_{c,sq} = 1/2$. The value of $v_{c,tri}$ for the triangular lattice is determined as the (unique) real positive solution of the equation [8] $v^3 + 3v^2 - q = 0$. Solving this for $q = 1$ and subsituting into eq. (2.6) yields the expression above for $p_{c,tri}$. From the duality of the triangular and honeycomb lattices, one obtains the critical equation for the latter from the former by the inversion map $v/\sqrt{q} \to \sqrt{q}/v$, which yields $v^3 - 3qv - q^2 = 0$. Again setting $q = 1$, calculating the unique positive root of this equation, and substituting into (2.6) gives the expression for $p_{c,hc}$.

We next comment on p_c for the kagomé lattice and first recall the definition of this lattice. An Archimedean lattice is a uniform tiling of the plane by one or more regular polygons with the property that every vertex is equivalent to every other vertex. An Archimedean lattice is specified by the ordered product $\prod p_i^{a_i}$, where, as one makes a small circuit (say clockwise) around any vertex, one traverses the sequence of polygons p_i, with the possibility that a given p_i occurs a_i times in a row (for notation, see, e.g., [38]). In this notation, the kagomé is defined as $(3 \cdot 6 \cdot 3 \cdot 6)$. Although the critical bond percolation probability $p_{c,kag}$ is not known exactly, it has been determined numerically to high precision [21] as $p_{c,kag} = 0.5244053(3)$, where the number in parentheses is the estimated uncertainty in the last digit.

B. Values of $\langle k \rangle_{c,\Lambda}$

Because $\langle k \rangle$ has only a finite nonanalyticity at $p = p_{c,\Lambda}$, one can obtain rough values of $\langle k \rangle_{c,\Lambda}$ from Taylor series

expansions around $p = 0$ and $p = 1$. Using results from [5, 6], Ref. [22] obtained explicit exact values of $\langle k \rangle_c$ for bond percolation on three two-dimensional lattices:

$$\langle k \rangle_{c,sq} = \frac{3^{3/2} - 5}{2} = 0.0980762... \qquad (4.1)$$

and

$$\langle k \rangle_{c,tri} = \frac{35}{4} - \frac{3}{p_{c,tri}} = 0.111844... \qquad (4.2)$$

and, using also [4],

$$\langle k \rangle_{c,hc} = \frac{1}{2}(\langle k \rangle_{c,tri} + p_{c,tri}^3) = 0.07686667... \qquad (4.3)$$

(Here we take into account that $\langle k \rangle$ is defined per site, while the quantity n_c^{B-HC} in [22] is defined per unit cell and there are two sites per unit cell on the honeycomb lattice.) Since $p_{c,kag}$ is not known exactly, neither is $\langle k \rangle_{c,kag}$. We are not aware of numerical simulations of bond percolation on kagomé lattices that have yielded an approximate value of $\langle k \rangle_{c,kag}$, or of reasonably long series expansions that could be used to obtain an accurate estimate of this quantity.

C. Series Expansions

Taylor series expansions played an important role in the early investigation of critical exponents and tests of scaling for percolation. These expansions were performed about the points $p = 0$ (low-density) and $p \to 1$ (high-density), or equivalently, $r \to 0$, where

$$r = 1 - p . \qquad (4.4)$$

These series expansions have typically been calculated for one of two related quantities, the average number, per site, of bond clusters, $\langle k_{bond} \rangle$, or the average number, per bond (pb), of bond clusters, $\langle k_{bond} \rangle_{pb}$. For an (infinite) lattice Λ with coordination number κ_Λ, there are $\kappa_\Lambda/2$ bonds per site, so $\langle k_{bond} \rangle = (\kappa_\Lambda/2)\langle k_{bond} \rangle_{pb}$. The average number of components (= site clusters) per site, $\langle k \rangle$, differs from the average number of bond clusters per site, $\langle k_{bond} \rangle$ in that $\langle k \rangle$ counts all components, including isolated sites, whereas $\langle k_{bond} \rangle$ counts the bond clusters that have a nonzero number of bonds and therefore excludes isolated sites. Hence, for an (infinite) lattice Λ,

$$\langle k \rangle_\Lambda = \langle k_{bond} \rangle_\Lambda + (1 - p)^{\kappa_\Lambda} . \qquad (4.5)$$

Clearly, $\langle k_{bond} \rangle \to 0$ rather than 1 as $p \to 0$, since no bond clusters with a nonzero number of bonds are present on G in this limit.

Early calculations of series for the average bond cluster number $\langle k_{bond} \rangle$ were carried out in Refs. [3, 4] and, to higher order in [9] for the square, triangular, and honeycomb lattices. These have the form

$$\langle k_{bond} \rangle = \sum_{s=1}^{\infty} p^s D_s(r) \qquad (4.6)$$

where r is given by eq. (4.4) and the D_s are the perimeter polynomials. Using the relation (4.5), it is straighforward to calculate series for $\langle k \rangle$ from those for $\langle k_{bond} \rangle$. For our present purposes, we shall need only the first few terms of these series. For $p \to 0$ these are

$$\langle k \rangle_{sq} = 1 - 2p + p^4 + 2p^6 - 2p^7 + 7p^8 + O(p^9) \quad (4.7)$$

$$\langle k \rangle_{tri} = 1 - 3p + 2p^3 + 3p^4 + 3p^5 + 3p^6 + 6p^7 + O(p^9) \quad (4.8)$$

$$\langle k \rangle_{hc} = 1 - \frac{3}{2}p + \frac{1}{2}p^6 + \frac{3}{2}p^{10} + O(p^{11}) \ . \qquad (4.9)$$

Thus, for small p, $\langle k \rangle$ is linear, with slope $-\kappa_\Lambda/2$, so that $\langle k \rangle$ decreases more rapidly on a lattice with larger coordination number.

For the $p \to 1$ series, we have

$$\langle k \rangle_{sq} = r^4 + 2r^6 - 2r^7 + 7r^8 + O(r^9) \qquad (4.10)$$

$$\langle k \rangle_{tri} = r^6 + 3r^{10} - 3r^{11} + 2r^{12} + O(r^{14}) \qquad (4.11)$$

$$\langle k \rangle_{hc} = r^3 + \frac{3}{2}r^4 + \frac{3}{2}r^5 + \frac{3}{2}r^6 + O(r^7) \ . \qquad (4.12)$$

These have the general form $\langle k \rangle_\Lambda = r^{\kappa_\Lambda} + ...$ as $p \to 1$, so that the larger the coordination number is, the smaller $\langle k \rangle$ in these expansions.

Because of the significant finite-width effects on the lattice strips considered here, one does not expect the Taylor series expansions of the exact expressions for $\langle k \rangle$ on these strips to match many orders of the expansions for the corresponding two-dimensional lattices. However, our exact results can give new insight into one property of these series. In early work it was found that the radii of convergence of Taylor series expansions around both $p = 0$ and $p = 1$ were typically set by unphysical singularities, and these radii of convergence were less than the distance from the expansion point to the physical singularity, $p_{c,\Lambda}$, for the small-p expansions and $r_{c\Lambda} = 1 - p_{c,\Lambda}$ for small-r expansions [9]. (This was also the case for three-dimensional lattices [10].) This is reminiscent of the situation for low-temperature Taylor series expansions for various discrete spin models such as the Ising model, whose radii of convergence in the expansion variable $z = e^{-\beta J}$ were often smaller than the distance from

the origin to the actual critical point z_c, again as a consequence of complex-temperature singularities [39, 40]. Although spin models with finite-range spin-spin interactions only have possible phase transitions at $T = 0$, the complex-temperature singularities of the free energy for such quasi-1D systems does exhibit some similarities with those of 2D systems [40]-[43]. These complex-temperature singularities are associated with complex-temperature phase boundaries $\mathcal{B}$, which are the continuous accumulation set of the (Fisher [44]) zeros of the partition function. In the same way, we can use our exact calculations of $\langle k \rangle$ for infinite-length, finite-width lattice strips to gain some insight into the unphysical singularities encountered in the above-mentioned series studies for percolation.

V. STRIPS OF THE SQUARE LATTICE

A. $L_y = 1$

The well-known result

$$\langle k \rangle_{1D} = 1 - p \qquad (5.1)$$

for the infinite line can be derived directly using probability methods. Here we illustrate how it can be derived via eq. (2.8). An elementary calculation yields the Potts free energy $f(1D, q, v) = \ln(q + v)$. Using eq. (2.8) yields the above result for $\langle k \rangle_{1D}$. This has the value $1/2$ at $p = p_{c,sq}$ (see Table I). Note that the bond cluster number per site in 1D is $\langle k_{bond} \rangle = p(1 - p)$, in accord with the relation (4.5).

B. Free Transverse Boundary Conditions

For $L_y \geq 2$, we label an infinite-length strip of width L_y of the lattice Λ with given transverse boundary conditions BC_y as $\Lambda, (L_y)_{BC_y}$. In particular, the $L_y = 2$ square-lattice strips with free (F) and periodic (P) transverse boundary conditions are denoted $sq, 2_F$ and $sq, 2_P$.

1. 2_F

The free energy of the Potts model for the $sq, 2_F$ strip is [27]

$$f(sq, 2_F, q, v) = \frac{1}{2} \ln \lambda_{sq,2F,1} \qquad (5.2)$$

where

$$\lambda_{sq,2F,j} = \frac{1}{2} \Big(T_{s2F} \pm \sqrt{R_{s2F}} \Big) \qquad (5.3)$$

with $j = 1, 2$ corresponding to $\pm$ and

$$T_{s2F} = v^3 + 4v^2 + 3qv + q^2 \qquad (5.4)$$

$$R_{s2F} = v^6 + 4v^5 - 2qv^4 - 2q^2v^3 + 12v^4$$
$$+ 16qv^3 + 13q^2v^2 + 6q^3v + q^4 . \qquad (5.5)$$

In eq. (5.3) only the $j = 1$ term is relevant for the free energy, while the $j = 2$ term will be discussed below.

From eq. (5.2) we calculate the average cluster number per site

$$\langle k \rangle_{sq,2_F} = \frac{(1-p)^2(2+p-2p^2)}{2(1-p^2+p^3)} . \qquad (5.6)$$

This is plotted in Fig. 1 together with cluster numbers calculated for other strips. At $p = p_{c,sq} = 1/2$, this average cluster number has the value $\langle k \rangle_{sq,2_F} = 2/7 \simeq 0.28571$.

From the exact expression for $\langle k \rangle_{sq,2_F}$ we compute the respective Taylor series expansions

$$\langle k \rangle_{sq,2_F} = 1 - \frac{3}{2}p + \frac{1}{2}p^4 + \frac{1}{2}p^6 + O(p^7) \quad \text{for} \quad p \to 0 \quad (5.7)$$

and, in terms of the variable r in eq. (4.4),

$$\langle k \rangle_{sq,2_F} = \frac{1}{2}r^2 + 2r^3 - \frac{7}{2}r^5 + O(r^6) \quad \text{for} \quad r \to 0 . \quad (5.8)$$

Thus for this strip $\langle k \rangle$ is linear for small p and vanishes quadratically as $p \to 1$. As expected for such a small width, these series differ from the series for the square lattice, although the linear behavior for small p is common to both.

The expression for $\langle k \rangle$ for this strip has singularities, which are simple poles, where the denominator $1 - p^2 + p^3 = 0$, at

$$p \simeq -0.7549 , \quad 0.8774 \pm 0.7449i . \qquad (5.9)$$

The first of these poles is the closest to the origin and determines the radius of convergence of the small-p Taylor series in eq. (5.7) to be approximately 0.7549. The complex pair are the same distance from the point $p = 1$ and imply that this series converges for $|1 - p| \lesssim 0.7549$. Thus, although $\langle k \rangle_{sq,2_F}$ is an analytic function of p for $p \in [0, 1]$, the Taylor series expansions about $p = 0$ and $p = 1$ have radii of convergence less than unity because of singularities of this function at real and complex values outside the physical interval $0 \le p \le 1$. It is interesting that although $\lambda_{sq,2_F}$ is an algebraic function of v (hence p, via eq. (2.6)), the resultant expression for $\langle k \rangle_{sq,2_F}$ is a rational function of p. However, this is a consequence of the small value of L_y. The same comment applies to the

property that $\langle k \rangle_{sq,2_F}$ is meromorphic, i.e., its only singularities are simple poles. As will be seen, these features are also true of the cluster number $\langle k \rangle$ for other $L_y = 2$ strips considered here.

In this very simple context of a quasi-1D strip, one hence gains some insight into the similar influence of unphysical singularities in series expansions about $p = 0$ and $p = 1$ for percolation on higher-dimensional lattices. To understand these poles more deeply, we observe that although the free energy $f(sq, 2_F, q, v)$ only depends on $\lambda_{sq,2F,1}$, the partition function for free longitudinal boundary conditions [45] and $q \ne 1$ in general is a symmetric sum of L_x'th powers of both of the $\lambda_{sq,2F,j}$'s for $j = 1$ and $j = 2$ (given as eq. (5.17) of Ref. [27]). We are interested in the limit $q \to 1$. It is necessary to take account of a subtlety concerning the dependence of the complex-v phase boundary $\mathcal{B}$ of the Potts model, as a function of q. In previous work (see eqs. (2.8)-(2.12) of Ref. [27], eq. (1.9) of Ref. [46])) we have pointed out the noncommutativity at certain special values of q_s, including $q_s = 0, 1$, namely

$$\lim_{n \to \infty} \lim_{q \to q_s} Z(G, q, v)^{1/n} \ne \lim_{q \to q_s} \lim_{n \to \infty} Z(G, q, v)^{1/n} \qquad (5.10)$$

and we have noted that because of this noncommutativity, for the special set of points $q = q_s$ one must distinguish between (i) $(\mathcal{B}(\{G\}, q_s))_{nq}$, the continuous accumulation set of the zeros of $Z(G, q, v)$ obtained by first setting $q = q_s$ and then taking $n \to \infty$, and (ii) $(\mathcal{B}(\{G\}, q_s))_{qn}$, the continuous accumulation set of the zeros of $Z(G, q, v)$ obtained by first taking $n \to \infty$, and then taking $q \to q_s$. For these special points,

$$(\mathcal{B}(\{G\}, q_s))_{nq} \ne (\mathcal{B}(\{G\}, q_s))_{qn} . \qquad (5.11)$$

A previous case of this was the $q = 2$ (Ising) special case of the Potts model. Indeed, in that case it was noted that $\mathcal{B}_{qn}$ does not have the inversion symmetry $e^K \to e^{-K}$ that characterizes the Ising model and its complex-temperature phase boundary $\mathcal{B}_{nq}$ for a bipartite lattice (see pp. 396, 433-435 of Ref. [27]). This noncommutativity is also present at the value $q = 1$ relevant for percolation. If one uses the definition $\mathcal{B}_{nq}$ with $q = 1$ for percolation, as one uses the definition $\mathcal{B}_{nq}$ with $q = 2$ for the Ising model, then while $\mathcal{B}_{nq}$ is nontrivial for the Ising model, $\mathcal{B}_{nq}$ is trivial for the percolation problem. The reason for this is that if one sets $q = 1$ first, then, from the Hamiltonian definition of the Potts model, since the spins are the same on all sites, the spin-spin interactions on each bond contribute a factor e^K to the partition function, so one has the elementary result

$$Z(G, 1, v) = e^{K e(G)} = (v + 1)^{e(G)} \qquad (5.12)$$

Substituting $v = v_p$ as in eq. (2.6) gives

$$Z(G, 1, v_p) = (1 - p)^{-e(G)} \qquad (5.13)$$

Evidently, $Z(G, 1, v_p)$ has no zeros, so that $\mathcal{B}_{nq} = \emptyset$ in the complex p plane. Equivalently, $Z(G, 1, v)$ has only a single zero at the point $v = -1$, which maps, via eq. (2.6), to the circle at infinity in the complex p plane. We have noted above that in general, for $q \neq 1$, the partition function of the Potts model consists of a symmetric sum of L_x'th powers of $\lambda_{sq,2_F,1}$ and $\lambda_{sq,2_F,2}$; if one sets $q = 1$, the coefficient of $(\lambda_{sq,2_F,2})^{L_x}$ vanishes, and the $(\lambda_{sq,2_F,1})^{L_x}$ term, with its coefficient, reduces to the form (5.13) (where in the labelling convention of Ref. [27], $L_x + 1$ denotes the number of squares on the $sq, 2_F$ strip).

However, the fact that eq. (2.8) involves a derivative means that it is sensitive to properties of the Potts model in the neighborhood of the point $p = 1$ as well as at this point. This suggests that one consider the possible role of the locus $\mathcal{B}_{qn}$, although one must use caution in doing this because of the noncommutativity discussed above. Below we shall use the notation $\mathcal{B}_{qn}$ to mean specifically the boundary defined for $n \to \infty$ and $q \to 1$, relevant to the percolation problem.

We find some intriguing connections between the locus $\mathcal{B}_{qn}$ and complex-p singularities in $\langle k \rangle$. Let us first calculate $\mathcal{B}_{qn}$ for the $sq, 2_F$ strip as $q \to 1$ but with $q \neq 1$. Evaluating eq. (5.3) for $q \to 1$, we obtain

$$\lambda_{sq,2F,1} = \frac{1}{(1 - p)^3} \qquad (5.14)$$

and

$$\lambda_{sq,2F,2} = \frac{p^2}{(1 - p)^2} \ . \qquad (5.15)$$

The locus $\mathcal{B}_{qn}$ is the set of solutions of the equation of degeneracy in magnitude of dominant λ's. This locus can be seen as a special case of the more general phase boundary for the Potts model in the v plane for a fixed q, or in the q plane for fixed v. For the present case, since there are only two $\lambda_{sq,2_F,j}$'s, for $j = 1, 2$, this equation is $|\lambda_{sq,2F,1}| = |\lambda_{sq,2F,2}|$, i.e.,

$$|p^2(1 - p)| = 1 \ . \qquad (5.16)$$

In terms of the polar coordinates $p = \rho e^{i\theta}$ this equation reads $\rho^4(1 + \rho^2 - 2\rho \cos\theta) = 1$. The solution is a closed egg-shaped curve, shown in Fig. 2, that crosses the real p axis at $p \simeq -0.7549$ and $p \simeq 1.466$ and the imaginary p axis at $p \simeq \pm 0.8688i$. This thus constitutes the phase boundary in the complex p plane, separating this plane into two regions. As follows from the general discussion above, the physical interval $0 \leq p \leq 1$ lies entirely in one phase. The three poles of $\langle k \rangle_{sq,2_F}$ listed in eq. (5.9) lie on this boundary $\mathcal{B}_{qn}$.

2. $3_F, 4_F, 5_F$

The Potts model free energy f for the infinite-length $sq, 3_F$ strip was calculated in Ref. [28]. The free energy is given by $f(sq, 3_F, q, v) = (1/3) \ln \lambda_{sq,3_F}$, where $\lambda_{sq,3_F}$ is the (maximal) root of an algebraic equation of degree 4. Because of the complicated nature of the expression for this quartic root, we do not present it here. We have calculated $f(sq, 3_F, q, v)$, and hence $\langle k \rangle_{sq,3_F}$, to high precision by numerically solving for $f(sq, 3_F, q, v)$ for a range of values of q near unity, for each value of p, and carrying out the differentiation in eq. (2.8). Although this is numerical, the computational steps can be carried out with almost arbitrarily high precision, so that, in practice, it is essentially equivalent to evaluating an explicit exact analytic expression. We also apply this procedure for larger strip widths, using the exact calculation of f for $sq, 4_F$ and $sq, 5_F$ in Ref. [35] (see also [37]). The resulting values of $\langle k \rangle$ are plotted as functions of p in Fig. 1, and the values of $\langle k \rangle$ at $p = p_{c,sq} = 1/2$ are listed in Table I. One could carry out similar calculations of $\langle k \rangle$ for larger values of L_y, but our results are sufficient to show the nature of the approach of $\langle k \rangle$ on these infinite-length, finite-width strips to the average cluster number for the corresponding infinite two-dimensional lattice. Indeed, one of the most interesting pieces of information that we get from our results - the exact determination of singularities of $\langle k \rangle$ in the complex p plane and their effect on the radii of convergence of series expansions, can only be obtained for strip widths that are small enough so that we can get exact explicit analytical forms for $\langle k \rangle$.

C. Periodic Transverse Boundary Conditions

1. 2_P

By using periodic transverse boundary conditions, one minimizes finite-width effects in this transverse direction. We consider first the $sq, 2_P$ strip. Note that this strip has double transverse bonds. The free energy was computed in Ref. [28] and is given by

$$f(sq, 2_P, q, v) = \frac{1}{2} \ln \lambda_{sq,2P,1} \qquad (5.17)$$

where

$$\lambda_{sq,2P,j} = \frac{1}{2}\Big(T_{s2P} \pm \sqrt{R_{s2P}}\,\Big) \qquad (5.18)$$

with $j = 1, 2$ corresponding to $\pm$ and

$$T_{s2P} = 6v^2 + 4qv + q^2 + 4v^3 + qv^2 + v^4 \qquad (5.19)$$

and
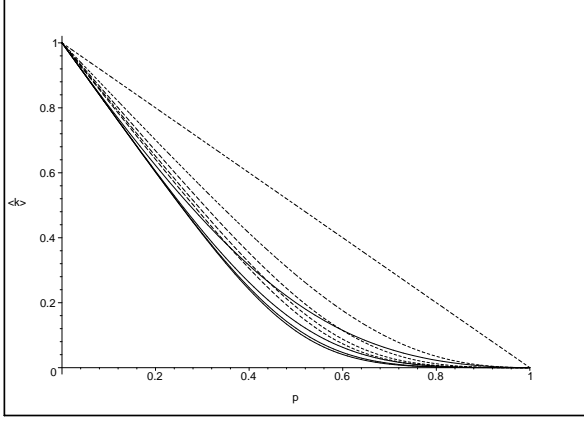
$$R_{s2P} \ = \ (v^4 + 6v^3 + 8v^2 + 3qv^2 + 6qv + q^2)$$

FIG. 1: Plots of $\langle k \rangle$ (vertical axis) as a function of $p \in [0,1]$ (horizontal axis) for infinite-length, finite-width strips of the square lattice. The dashed and solid curves refer to free and periodic transverse boundary conditions, respectively. For a given p, the dashed curves are, in order of descending value of $\langle k \rangle$, for $1_F \le (L_y)_F \le 5_F$, and the solid curves are, in the same order, for $2_P \le (L_y)_P \le 5_P$.
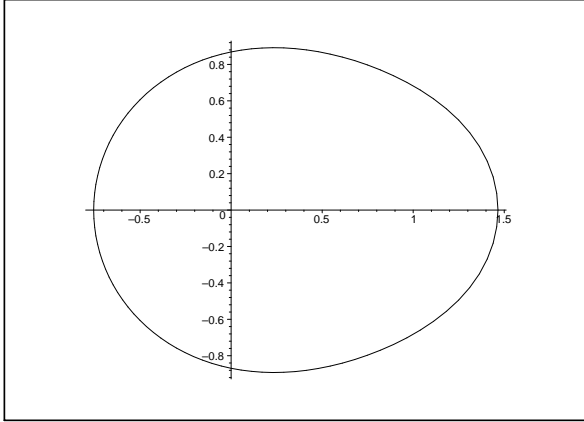

FIG. 2: Plot of the boundary $\mathcal{B}_{qn}$ in the complex p plane for the infinite-length $sq, 2_F$ lattice strip. Horizontal and vertical axes are $Re(p)$ and $Im(p)$.

$$\times \ (v^4 + 2v^3 + 4v^2 - qv^2 + 2qv + q^2) \ . \quad (5.20)$$

From this, using eq. (2.8), we calculate

$$\langle k \rangle_{sq,2_P} = \frac{(1-p)^2(2-3p^2+2p^3)}{2(1+p-p^2)(1-p+p^2)} \ . \quad (5.21)$$

At the value of $p_{c,sq} = 1/2$ for the infinite square lattice this has the value $\langle k \rangle_{sq,2_P} = 1/5$.

The expression (5.21) has poles where $1 + p - p^2$ vanishes, at

$$p_{1,2} = \frac{1}{2}(1 \pm \sqrt{5}\,) \simeq 1.618 \ , \quad -0.6180 \quad (5.22)$$

and where $1 - p + p^2$ vanishes, at

$$p_{3,4} = \frac{1}{2}(1 \pm \sqrt{3}\,i) \simeq 0.5 \pm 0.866i \ . \quad (5.23)$$

The second and first of these poles are closest to the points $p = 0$ and $p = 1$ and determine the radii of convergence of the respective Taylor series expansions about these points both to be 0.618. These expansions are

$$\langle k \rangle_{sq,2_P} = 1 - 2p + \frac{1}{2}p^2 + 2p^4 + O(p^5) \quad (5.24)$$

and

$$\langle k \rangle_{sq,2_P} = \frac{1}{2}r^2 + 2r^4 - 2r^5 + O(r^6) \ . \quad (5.25)$$

Note that, in accord with the fact that the coordination number of this and any infinite-length lattice strip of the square lattice with periodic transverse boundary conditions is 4, the coefficient of the linear term in the small-p expansion is equal to that of the expansion for the full square lattice.

We next discuss the connection of the poles in $\langle k \rangle_{sq,2_P}$ with the locus $\mathcal{B}_{qn}$. Although $f(sq, 2_P, q, v)$ only depends on the quantity $\lambda_{sq,2P,1}$, the Potts model partition function for $q \ne 1$ involves a symmetric sum of L_x'th powers of both $\lambda_{sq,2P,1}$ and $\lambda_{sq,2P,2}$ [35]. Evaluating $\lambda_{sq,2P,j}$ for $q \to 1$, we have

$$\lambda_{sq,2P,1} = \frac{1}{(1-p)^4} \quad (5.26)$$

$$\lambda_{sq,2P,2} = \frac{p^2}{(1-p)^2} \quad (5.27)$$

The locus $\mathcal{B}_{qn}$ is the set of solutions of the equation $|\lambda_{sq,2P,1}| = |\lambda_{sq,2P,2}|$, i.e.

$$|p(1-p)| = 1 \ . \quad (5.28)$$

In terms of the polar coordinates defined above, this equation reads $\rho^2(1 + \rho^2 - 2\rho \cos\theta) = 1$. The solution forms a closed oval curve in the complex p, shown in Fig. 3, that crosses the real axis at the points $p_{1,2}$ in eq. (5.22) and the imaginary axis at the points $p \simeq \pm 0.78615i$. As in the case of the $sq, 2_F$ strip, this curve separates the p plane into two regions. All of the four poles of $\langle k \rangle_{sq,2_P}$ given in eqs. (5.22) and (5.23) lie on this curve $\mathcal{B}_{qn}$. This property - that the singularities of $\langle k \rangle$ lie on $\mathcal{B}_{qn}$ -

is analogous to the property that the singularities of thermodynamic functions of spin models lie on the complex-temperature phase boundaries for these models, as we have studied in earlier work [40]-[43]. Having pointed out the connection between these singularities and the locus $\mathcal{B}_{qn}$, we shall, for the strips considered below, just summarize the singularities of $\langle k \rangle$.
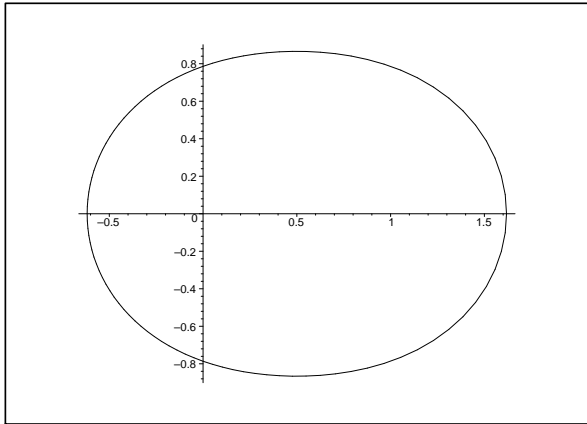

FIG. 3: Plot of the boundary $\mathcal{B}_{qn}$ in the complex p plane for the infinite-length $sq, 2_P$ lattice strip. Horizontal and vertical axes are $Re(p)$ and $Im(p)$.

2. $3_P, 4_P, 5_P$

For the $sq, 3_P$ strip, $f(sq, 3_P, q, v) = (1/3) \ln \lambda_{sq,3_P}$, where $\lambda_{sq,3_P}$ is the (maximal) root of a cubic equation. Although it is possible to display an analytic result for $\langle k \rangle_{sq,3_P}$, it is sufficiently cumbersome that we do not give it here. It is an algebraic, rather than rational, function of p. We do display the small-p expansion, which is

$$\langle k \rangle_{sq,3P} = 1 - 2p + \frac{1}{3}p^3 + p^4 + O(p^5) \ . \qquad (5.29)$$

The free energy f was calculated for the $sq, 4_P$ and $sq, 5_P$ strips in Ref. [35], and $\lambda_{sq,4_P}$ and λ_{5_P} are roots of equations of too high a degree to allow an explicit analytic solution. Accordingly, we compute $\langle k \rangle$ by the numerical procedure discussed above. Results are given in Fig. 1 and Table I.

D. Self-Dual Strips of the Square Lattice

It is of interest to calculate $\langle k \rangle$ for strips of the square lattice that maintain a property of the infinite square lattice, namely self-duality. The strips with free and periodic transverse boundary conditions considered above are not self-dual. However, one can construct a cyclic strip that is self-dual by adding a single external site to a cyclic square-lattice strip of width L_y and then adding bonds connecting all of the sites on one side of the strip to this single external site. We denote a self-dual (sd) strip of this type as $sq, (L_y)_{sd}$. Before presenting our calculations, a remark is in order concerning p_c for these strips. The physical meaning of p_c for a usual infinite lattice is, as mentioned before, that for $p \geq p_c$ there exists an percolation cluster linking two points that are an arbitrarily large distance apart. Now consider the simplest of the cyclic self-dual lattice graphs, with $L_y = 1$; this is a wheel graph, having a rim forming a circuit and a central site ($\sim$ axle) connected to the sites on the rim by L_x bonds forming spokes. Evidently, even in the limit $L_x \to \infty$, the maximum distance between any two sites on this lattice graph is 2 bonds; to get from any site on the rim to any other site, one takes a minimum-distance route that goes inward along one spoke to the central site and out again on another spoke to the other site. Similarly, for any finite L_y, even as $L_x \to \infty$ there is a maximal finite distance $2L_y$ bonds between any two sites. Therefore, although this family of cyclic lattice strips does maintain the property of self-duality of the infinite square lattice, the notion of a critical p_c beyond which there is a percolation cluster linking two sites arbitrarily apart is not applicable to it since no sites are arbitrarily far apart.

1. 1_{sd}

The free energy is [31, 34]

$$f(sq, 1_{sd}, q, v) = \ln \lambda_{sd1} \qquad (5.30)$$

where

$$\lambda_{sd1,j} = \frac{1}{2}(T_{sd1} + \sqrt{R_{sd1}} \ ) \qquad (5.31)$$

with

$$T_{sd1} = 3v + q + v^2 \qquad (5.32)$$

$$R_{sd1} = 5v^2 + 2vq + 2v^3 + q^2 - 2v^2q + v^4 \ . \qquad (5.33)$$

From this we calculate

$$\langle k \rangle_{sq,1_{sd}} = \frac{(1-p)^3}{1 - p + p^2} \ . \qquad (5.34)$$

We have $\langle k \rangle_{sq,1_{sd}} = 1/6$ at $p = p_{c,sq}$. The mean cluster number $\langle k \rangle$ in eq. (5.34) has the following Taylor series expansions for $p \to 0$ and $p \to 1$:

$$\langle k \rangle_{sq,1_{sd}} = 1 - 2p + p^3 + p^4 - p^6 - p^7 + p^9 + O(p^{10}) \quad (5.35)$$

$$\langle k\rangle_{sq,1_{sd}} = r^3 + r^4 - r^6 - r^7 + r^9 + O(r^{10}) \ . \quad (5.36)$$

One sees that the coefficient of the linear term in the small-p expansion correctly matches that of the series for the infinite square lattice and the power of the leading-order term in $p \to 1$ expansion is 3, which, although not equal to the power 4 in the corresponding expansion in eq. (4.10), is at least closer than the power of 2 for the $L_y = 2$ square-lattice strips with free or periodic boundary conditions. The poles in eq. (5.34) at $p = (1/2)(1 \pm \sqrt{3}i)$ set the radii of convergence of the small-p and small-r expansions as unity in both cases, i.e., the full physical interval $0 \le p \le 1$.

2. $2_{sd}, 3_{sd}, 4_{sd}$

For these strips, the free energy has the form [31, 34] $f(sq, (L_y)_{sd}, q, v) = (1/L_y) \ln \lambda_{sq,(L_y)_{sd}}$, where $\lambda_{sq,(L_y)_{sd}}$ are maximal roots of algebraic equationa of degree 5 or higher. Hence, it is thus not possible to obtain a closed-form analytic solution for this root. We thus follow the same high-precision numerical procedure as described above.
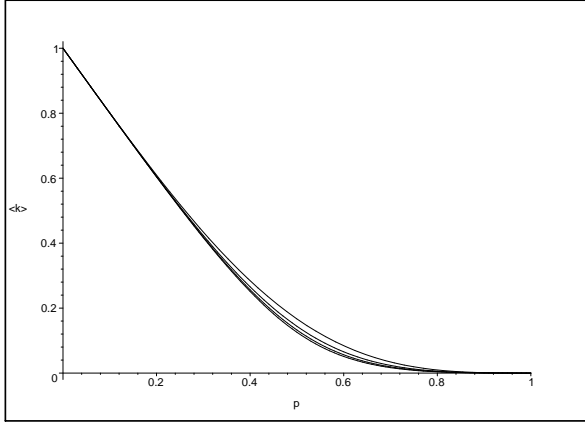

FIG. 4: Plots of $\langle k\rangle$, as a function of $p \in [0, 1]$, for infinite-length, finite-width self-dual strips of the square lattice. For a given p, in order of descending value of $\langle k\rangle$, the curves refer to are for $1_{sd} \le (L_y)_{sd} \le 4_{sd}$.

VI. STRIPS OF THE TRIANGULAR LATTICE

A. Free Transverse Boundary Conditions

1. 2_F

The free energy for the Potts model on this strip is [29]

$$f(tri, 2_F, q, v) = \frac{1}{2} \ln \lambda_{t2F} \quad (6.1)$$

where

$$\lambda_{t2F} = \frac{1}{2}\left[T_{t2F} + (3v + v^2 + q)\sqrt{R_{t2F}} \right] \quad (6.2)$$

with

$$T_{t2F} = v^4 + 4v^3 + 7v^2 + 4qv + q^2 \quad (6.3)$$

and

$$R_{t2F} = q^2 + 2qv - 2qv^2 + 5v^2 + 2v^3 + v^4 \ . \quad (6.4)$$

From this we calculate

$$\langle k\rangle_{tri,2_F} = \frac{(1-p)^3}{1 - p + p^2} \ . \quad (6.5)$$

Note that this expression for $\langle k\rangle$ is the same as that for the $L_y = 1$ self-dual strip in eq. (5.34). This provides an illustration of the fact that two different families of lattice strips may have the same average cluster number $\langle k\rangle$. We plot this cluster number $\langle k\rangle$ in Fig. 5, together with the cluster numbers for the various strips of the triangular lattice with greater widths and free or periodic boundary conditions. The values of $\langle k\rangle$ for $p = p_{c,tri}$ are listed in Table I. The Taylor series expansions of eq. (6.5) for $p \to 0$ and $r = 1 - p \to 0$ are the same as those of (5.34).

2. $3_F, 4_F, 5_F$

The free energy f for the strips of the triangular lattice with width $L_y = 3, 4, 5$ and free transverse boundary conditions were computed in Ref. [36] (see also [37]). We have used these exact analytic expressions to obtain high-precision numerical computations of $\langle k\rangle$ for these strips.

B. Periodic Transverse Boundary Conditions

1. 2_P

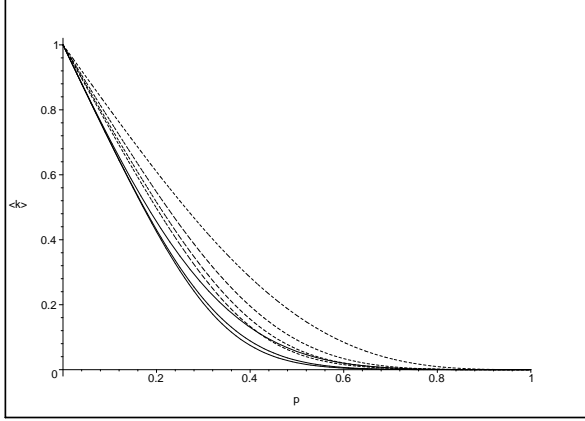Having explained our calculational method above for the square-lattice and previous triangular-lattice strips,

FIG. 5: Plots of $\langle k \rangle$, as a function of $p \in [0,1]$, for infinite-length, finite-width strips of the triangular lattice. The dashed and solid curves refer to free and periodic transverse boundary conditions, respectively. For a given p, the dashed curves are, in order of descending value of $\langle k \rangle$, for $2_F \leq (L_y)_F \leq 5_F$, and the solid curves are, in the same order, for $2_P \leq (L_y)_P \leq 4_P$.

we omit the details for other lattice strips except where we have carried out new calculations of Potts model free energies. The free energy $f(tri, 2_P, q, v)$ was calculated in Ref. [29]. From it we compute

$$\langle k \rangle_{tri,2_P} = \frac{(1-p)^4(2 + 2p - 7p^2 + 4p^3 - p^4 + 2p^5 - p^6)}{2(1 - 2p^2 + 8p^3 - 12p^4 + 8p^5 - 2p^6)}$$
(6.6)

This has the respective Taylor series expansions for $p \to 0$ and $p \to 1$:

$$\langle k \rangle_{tri,2_P} = 1 - 3p + \frac{1}{2}p^2 + 4p^3 + \frac{9}{2}p^4 - 10p^5$$
$$- 10p^6 + O(p^7)$$
(6.7)

$$\langle k \rangle_{tri,2_P} = \frac{1}{2}r^4 + 2r^6 - 2r^8 + \frac{9}{2}r^{10} + O(r^{11}) . \quad (6.8)$$

The poles of $\langle k \rangle$ occur at

$$p \simeq -0.3744 , \quad 1.6539 ,$$

$$0.1731 \pm 0.6306i , \quad 1.1872 \pm 0.6924i . \ (6.9)$$

The first two poles, lying on the real p axis, are closest to the points $p = 0$ and $p = 1$ and determine the radii of convergence of the series about these points to be approximately 0.3744 and 0.6539, respectively.

Our procedure for strips with greater widths $L_y \geq 3$ is as before for the square lattice and $(L_y)_F$ triangular-lattice strips. The small-p series for the $tri, 3_P$ lattice is

$$\langle k \rangle_{tri,3_P} = 1 - 3p + \frac{7}{3}p^3 + 6p^4 + O(p^5) . \quad (6.10)$$

VII. STRIPS OF THE HONEYCOMB LATTICE

The free energy $f(hc, 2_F, q, v)$ was calculated in Ref. [30]. From it we obtain

$$\langle k \rangle_{hc,2_F} = \frac{(1-p)^2(4 + 3p + 2p^2 + p^3 - 4p^4)}{4(1 - p^4 + p^5)} . \quad (7.1)$$

This has the respective Taylor series expansions

$$\langle k \rangle_{hc,2_F} = 1 - \frac{5}{4}p + \frac{1}{4}p^6 + \frac{1}{4}p^{10} - \frac{1}{4}p^{11} + O(p^{14}) \ (7.2)$$

$$\langle k \rangle_{hc,2_F} = \frac{3}{2}r^2 + 3r^3 + O(r^4) . \quad (7.3)$$

The expression (7.1) has poles at

$$p \simeq -0.8567 , \quad -0.15005 \pm 0.8975i , \quad 1.0784 \pm 0.4969i .$$
(7.4)

The first of these is the nearest to the point $p = 0$, so that the small-p Taylor series converges for $|p| \lesssim 0.8567$. The last pair of complex-conjugate poles is closest to $p = 1$, so that the series for $r \to 0$ converges for $|1-p| \lesssim 0.5031$.

Strips of the honeycomb lattice with other widths and boundary conditions are analyzed using the same techniques as discussed above. The resulting cluster numbers $\langle k \rangle$ are plotted in Fig. 6 and the values for $p = p_{c,hc}$ are listed in Table I.

VIII. STRIPS OF THE KAGOMÉ LATTICE

A. 2_F

For the purpose of obtaining $\langle k \rangle$, we have carried out a calculation of the free energy of the Potts model on the 2_F strip of the kagomé lattice. The results are sufficiently lengthy that we list them in the appendix. From these we calculate

$$\langle k \rangle_{kag,2_F} = \frac{N_{k2F}}{D_{k2F}} \quad (8.1)$$

where
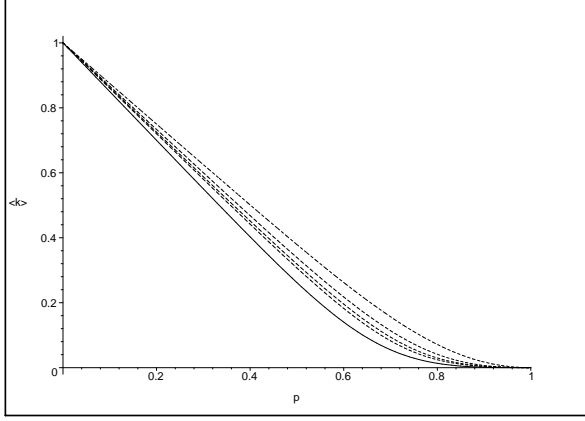
$$N_{k2F} = (1-p)^2(5 + 2p - p^2 - 2p^3 - 8p^4 - 16p^5$$

FIG. 6: Plots of $\langle k \rangle$, as a function of $p \in [0,1]$, for infinite-length, finite-width strips of the honeycomb lattice. The dashed and solid curves refer to free and periodic transverse boundary conditions, respectively. For a given p, the dashed curves are, in order of descending value of $\langle k \rangle$, for $2_F \leq (L_y)_F \leq 5_F$, and the solid curve is for 4_P.
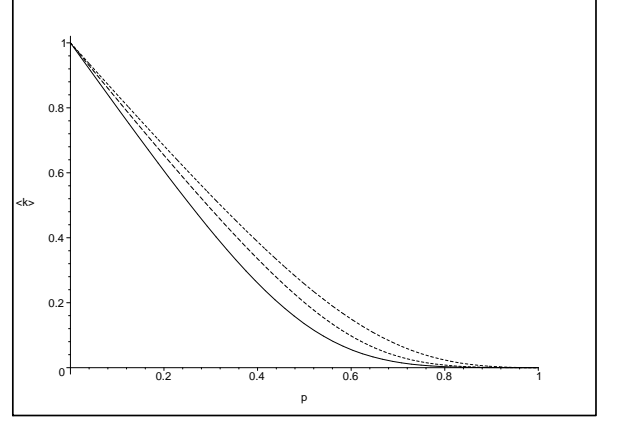

FIG. 7: Plots of $\langle k \rangle$, as a function of $p \in [0,1]$, for infinite-length, finite-width strips of the kagomé lattice. The dashed and solid curves refer to free and periodic transverse boundary conditions, respectively. For a given p, the dashed curves are, in order of descending value of $\langle k \rangle$, for $2_F \leq (L_y)_F \leq 3_F$, and the solid curve is for 2_P.

$$+43p^6 - 26p^7 - 2p^8 + 10p^9 - 3p^{10}$$

$$-2p^{11} + p^{12}) \tag{8.2}$$

and

$$D_{k2F} = 5(1 - p^4 - 2p^5 + 10p^6 - 10p^7 + 3p^8) . \tag{8.3}$$

We plot $\langle k \rangle_{kag,2F}$ in Fig. 7. At the value $p = p_{c,kag}$, $\langle k \rangle_{kag,2F}$ has the approximate value 0.22918. The $\langle k \rangle$ for this 2_F Kagomé strip has the following expansions in the vicinity of $p = 0$ and $p = 1$:

$$\langle k \rangle_{kag,2F} = 1 - \frac{8}{5}p + \frac{2}{5}p^3 + \frac{1}{5}p^6 + O(p^7) \tag{8.4}$$

$$\langle k \rangle_{kag,2F} = \frac{1}{5}r^2 + \frac{8}{5}r^3 + \frac{11}{5}r^4 - \frac{4}{5}r^5 + O(r^6) . \tag{8.5}$$

The cluster number (8.1) has poles at

$$p \simeq -0.5470 \pm 0.2862i , \quad -0.0363 \pm 0.6583i ,$$

$$0.7772 \pm 0.5605i , \quad 1.4728 \pm 0.1486i \tag{8.6}$$

Of these, the first and last complex-conjugate pairs are closest to $p = 0$ and $p = 1$, respectively, and determine the radii of convergence of the Taylor series expansions about these points to be approximately 0.6174 and 0.4956.

We have also calculated $\langle k \rangle$ for the 3_F strip of the kagomé lattice; this is plotted in Fig. 7.

B. 2_P

For the 2_P strip of the kagomé lattice we find

$$\langle k \rangle_{kag,2P} = \frac{N_{k2P}}{D_{k2P}} \tag{8.7}$$

where

$$N_{k2P} = (1-p)^4(6 + 12p + 12p^2 + 4p^3 - 25p^4 - 108p^5$$

$$+16p^6 + 472p^7 - 706p^8 + 320p^9 + 286p^{10} - 352p^{11}$$

$$-194p^{12} + 360p^{13} + 120p^{14} - 340p^{15} + 65p^{16} + 136p^{17}$$

$$-96p^{18} + 24p^{19} - 2p^{20}) \tag{8.8}$$

and

$$D_{k2P} = 6(1 - 2p^4 - 8p^5 + 32p^6 + 40p^7 - 268p^8$$

$$+424p^9 - 320p^{10} + 120p^{11} - 18p^{12}) . \tag{8.9}$$

A plot is given in Fig. 7. The cluster number $\langle k \rangle_{kag,2P}$ has poles at

$$p \simeq -0.4660 , \quad 1.6556 , \quad -0.3443 \pm 0.2919i ,$$

$$-0.0057 \pm 0.4751i , \quad 0.5325 \pm 0.48455i ,$$

$$1.0776 \pm 0.4384i , \quad 1.4785 \pm 0.2140i \tag{8.10}$$

The first complex-conjugate pair is the nearest to the origin and sets the radius of convergence of the small-p Taylor series expansion of $\langle k \rangle_{kag,2_P}$ as 0.4514, while the second-to-last complex-conjugate pair is closest to the point $p = 1$ and determines the radius of convergence of the series expansion about this point to be 0.4453, to the stated accuracy.

To our knowledge, it is not known what the value of $\langle k \rangle$ is for the (infinite) kagomé lattice at the numerically determined critical percolation probability $p_{c,kag}$. Assuming that, for a given set of transverse boundary conditions and a given $p \in (0, 1)$, $\langle k \rangle$ is a monotonically decreasing function of the strip width L_y for this lattice, as we find for other lattice strips, our results yield the upper bound $\langle k \rangle_{kag} < \langle k \rangle_{kag,2_P} \simeq 0.11149$ at the value $p = p_{c,kag}$ given above. Here we use the result for the 2_P strip since it is lower than the result for the 2_F and 3_F strips.

IX. DISCUSSION

We first introduce a notion of effective coordination number. For a graph G the degree of a vertex is the number of bonds connected to this vertex. A κ-regular graph is a graph in which all of the vertices have the same degree, κ. Whether a given lattice strip graph is κ-regular depends on the longitudinal and transverse boundary conditions; for example, it is κ regular if one uses toroidal (doubly periodic) boundary conditions. In the limit $L_x \to \infty$, since the longitudinal boundary conditions do not affect the free energy $f(\{G\}, q, v)$, we need only consider the effect of the transverse boundary conditions. The effective coordination number is

$$\kappa_{eff}(\{G\}) = \lim_{n \to \infty} \frac{2e(G)}{n(G)} . \quad (9.1)$$

Clearly $\kappa_{eff} = \kappa$ for a regular lattice. For regular lattice strips with periodic transverse boundary conditions, the value of κ_{eff} is the same as the value for the corresponding two-dimensional lattice. For strips with free transverse boundary conditions, we have

$$\kappa_{eff}(\Lambda, (L_y)_F) = \kappa_\Lambda \left(1 - \frac{\alpha}{L_y} \right) \quad (9.2)$$

where $\kappa_\Lambda = 4, 6, 3$ for $\Lambda = sq, tri, hc$ and

$$\alpha_{sq} = \frac{1}{2} , \alpha_{tri} = \frac{2}{3} , \quad \alpha_{hc} = \frac{1}{3} . \quad (9.3)$$

For the cyclic self-dual strips of the square lattice, the single external vertex connected to each of the sites on one side of the strip has a degree L_x that diverges as $L_x \to \infty$. The $L_x(L_y - 1)$ interior vertices have degree 4,

TABLE I: Values of $\langle k \rangle$ on infinite-length strips of the lattices Λ (where sq, tri, hc, kag denote square, triangular, honeycomb, and kagomé) of finite width L_y at $p = p_{c,\Lambda}$, The transverse boundary conditions (BC_y) are F and P for free and periodic, respectively. We also include results for the self-dual (sd) strips of the square lattice. The effective coordination number κ_{eff} is defined in eq. (9.1). The $L_y = \infty$ values of $\langle k \rangle_{p=p_{c,\Lambda}}$ are the values for the two-dimensional lattice Λ, and the dashes for these entries indicate that they do not depend on BC_y.

Λ	BC_y	L_y	κ_{eff}	$\langle k \rangle_{p=p_{c,\Lambda}}$
sq	F	1	2	0.50000
sq	F	2	3	0.28571
sq	F	3	3.33	0.21940
sq	F	4	3.50	0.18753
sq	F	5	3.60	0.16887
sq	P	2	4	0.20000
sq	P	3	4	0.14103
sq	P	4	4	0.12150
sq	P	5	4	0.11284
sq	sd	1	4	0.16667
sq	sd	2	4	0.14407
sq	sd	3	4	0.132545
sq	sd	4	4	0.12561
sq	−	∞	4	0.09808
tri	F	2	4	0.35958
tri	F	3	4.67	0.27149
tri	F	4	5	0.22946
tri	F	5	5.20	0.20491
tri	P	2	6	0.19091
tri	P	3	6	0.14665
tri	P	4	6	0.13138
tri	−	∞	6	0.11184
hc	F	2	2.50	0.20475
hc	F	3	2.67	0.16000
hc	F	4	2.75	0.13834
hc	F	5	2.80	0.12560
hc	P	4	3	0.08983
hc	−	∞	3	0.07687
kag	F	2	3.2	0.22918
kag	F	3	3.5	0.17220
kag	P	2	4	0.11149

while the L_x vertices on the rim have degree 3. Together, these lead, in the limit $L_x \to \infty$, to the result $\kappa_{sq,sd} = 4$. Finally, for the kagomé strips with free transverse boundary conditions

$$\kappa_{eff}(kag, (L_y)_F) = 4 \left(1 - \frac{1}{3L_y - 1} \right) \quad (9.4)$$

while for the kagomé strips with periodic transverse boundary conditions, $\kappa_{eff} = 4$, the same value as for the infinite two-dimensional kagomé lattice.

From our calculations we find a number of generic features:

- We have shown that $\langle k \rangle$ is a (real) analytic function of p in the interval $0 \leq p < 1$. At the critical percolation probability $p = 1$ for these quasi-1D strips, our exact results for $\langle k \rangle$ are also analytic, although some other quantities in percolation, such as the percolation probability $P(p)$ and the cluster size $S(p)$ are not, as is evident from the well-known 1D case.

- As the curves in the figures show, with an increase in strip width L_y, $\langle k \rangle$ is consistent with approaching a limiting function of p. This is in accord with one's expectation.

- For a given p in the interval between 0 and 1, and for a given type of lattice strip, as the width L_y increases, $\langle k \rangle$ decreases, so that the approach to the asymptotic value for the 2D lattice is from above, in the cases that we have computed. For strips with free transverse boundary conditions, increasing L_y increases κ_{eff}, so the decrease of $\langle k \rangle$ is associated with an increase in the effective coordination number. This is reasonable, since, heuristically, for a fixed value of p, there is a greater probability of having a percolating cluster on a lattice of higher coordination number, so that more sites are part of this cluster and there are fewer separate clusters per site. This is also reflected in the monotonic decrease of $p_{c,\Lambda}$ with increasing κ_Λ for most higher-dimensional lattices. (However, we recall that counterexamples to this general monotonic decrease of $\langle k \rangle$ with increasing coordination number are known [23, 26].)

  For strips with periodic transverse boundary conditions, the decrease of $\langle k \rangle$ at a fixed p with increasing width L_y is not associated with an increase in κ_{eff}, since κ_{eff} is constant for these strips (and equal to the two-dimensional value); here one may interpret the decrease as being simply due to a reduction in the finite-width effects that enables the percolation quantities to approach their two-dimensional values.

- For a given lattice type, we find some examples where the curve for $\langle k \rangle$ calculated on a strip of width L_y with periodic transverse boundary conditions will cross the curve for $\langle k \rangle$ for the same lattice and a different L_y and free transverse bound-

ary conditions. For example, as is evident in Fig. 1, the curve for $\langle k \rangle$ on the $sq, 2_P$ strip lies below those for $\langle k \rangle$ on the $sq, (L_y)_F$ strips at small p, but sequentially crosses the latter as p increases and lies above them (except for $L_y = 1, 2$) as $p \to 1^-$. Similar behavior is observed, e.g., on the strips of the triangular lattice. These also constitute examples of how $\langle k \rangle$ calculated on a strip with a larger value of κ_{eff} than that of another strip can be larger than $\langle k \rangle$ for the latter strip. For instance, $\kappa_{eff} = \kappa = 4$ for the $sq, 2_P$ strip, which is larger than the value $\kappa_{eff} = 3.6$ for the $sq, 5_F$ strip; however, $\langle k \rangle$ on the former strip is larger than $\langle k \rangle$ on the latter for $p \gtrsim 0.36$. This dependence on transverse boundary conditions is consistent with disappearing as the strip width $L_y \to \infty$, consistent with the approach to a single limiting function $\langle k \rangle$ for the corresponding 2D lattice. Although we have not proved rigorously that the function $\langle k \rangle$ obtained via this limiting sequence (taking $L_x \to \infty$ first and then taking $L_y \to \infty$) is identical to the function $\langle k \rangle$ obtained via the usual two-dimensional thermodynamic limit ($L_x \to \infty$, $L_y \to \infty$ with L_y/L_x a nonzero finite number), this conclusion is consistent with our findings.

- We have used the values of $\langle k \rangle$ at $p = p_{c,\Lambda}$ as a measure of how rapidly, for a given p, the cluster number calculated on infinite-length, finite-width strips approaches the value for the two-dimensional lattice. These values are listed in Table I. Even for the modest strip widths considered here, one sees that (i) these values approach the known values of $\langle k \rangle$ on the corresponding two-dimensional lattices reasonably quickly, and (ii) this approach is more rapid when one uses periodic transverse boundary conditions, as is expected, since the latter minimize finite-width effects. For example, for the strip of the square lattice with $L_y = 5$ and periodic transverse boundary conditions, $\langle k \rangle$ evaluated at $p = p_{c,sq}$ is about 15 % larger than the value (4.1) for the square lattice, while $\langle k \rangle$ for the $tri, 4_P$ and $hc, 4_P$ strips, evaluated at the respective $p_{c,tri}$ and $p_{c,hc}$, are both about 17 % larger than the corresponding values (4.2) and (4.3) for the triangular and honeycomb lattices.

- We find that for these strips, the small-p series expansions of $\langle k \rangle$ have the leading terms

$$\langle k \rangle = 1 - \left(\frac{\kappa_{eff}}{2} \right) p + ... \qquad (9.5)$$

which are analogous to the structure that these series have for regular lattices of dimension $d \geq 2$.

Higher-order terms in the series for the strips of small widths are not expected to coincide with those in the series for the two-dimensional lattices, and one sees that they do not.

- An interesting output of our analysis is the exact determination, for various infinite-length, finite-width strips, of the singularities of $\langle k \rangle$ in the complex p plane. As we have shown, for many strips these (real and/or complex) singularities outside the physical interval $[0,1]$ occur sufficiently close to the points $p = 0$ and $p = 1$ that they render the radii of convergence of the respective Taylor series expansions about these points less than unity, although the actual functions $\langle k \rangle$ themselves are analytic functions on $p \in [0,1]$. Although the strip widths are probably too small to justify a detailed comparison with unphysical singularities for percolation quantities in two dimensions, this generic property - the presence of unphysical singularities that determine the radii of the Taylor series expansions about the points $p = 0$ and $p = 1$ to be less than p_c for the given type of lattice - is similar to what was found in analyses of series for the percolation problem on two-dimensional lattices [9] (and three-dimensional lattices [10]).

- Finally, we have discussed how, for a given infinite-length, finite-width strip, the unphysical singularities have a connection with the locus $\mathcal{B}_{qn}$, which is the continuous accumulation set of the zeros of the Potts model partition function in the p (or equivalently the v) plane obtained by first letting $n \to \infty$ and then $q \to 1$. In particular, we find that these unphysical singularities lie on $\mathcal{B}_{qn}$. The noncommutativity of eq. (5.11) analyzed in Ref. [27] plays a crucial role here, since $\mathcal{B}_{nq}$, obtained by first letting $q \to 1$ and then $n \to \infty$, is trivial. Our results motivate further study on this topic.

X. CONCLUSIONS

In summary, we have presented exact calculations of the average cluster number per site $\langle k \rangle$ for the bond percolation problem on infinite-length, finite-width strips of the square, triangular, honeycomb, and kagomé lattices, with both free and periodic transverse boundary conditions. We believe that these results are a useful extension beyond the one-dimensional result toward two dimensions and provide insight into the form of $\langle k \rangle$ as a function of the bond occupation probability p.

XI. APPENDIX

In this appendix we give the free energy for the Potts model on the 2_F strip of the kagomé lattice. We find

$$f(kag, 2_F, q, v) = \frac{1}{5} \ln \lambda_{k2F} \tag{11.1}$$

where

$$\lambda_{k2F} = \frac{1}{2}[T_{k2F} + \sqrt{R_{k2F}}\,] \tag{11.2}$$

with

$$T_{k2F} = v^8 + 8v^7 + v^6 q + 29v^6 + 20v^5 q + 10v^4 q^2$$
$$+ \; 2v^3 q^3 + 42v^5 + 61v^4 q + 54v^3 q^2 + 28v^2 q^3$$
$$+ \; 8vq^4 + q^5 \tag{11.3}$$

and

$$R_{k2F} = v^{16} + 16v^{15} + 2v^{14}q + 114v^{14} + 32v^{13}q$$
$$-3v^{12}q^2 - 4v^{11}q^3 + 484v^{13} + 288v^{12}q + 52v^{11}q^2$$
$$-28v^{10}q^3 - 12v^9 q^4 - 2v^8 q^5 + 1329v^{12} + 1572v^{11}q$$
$$+1098v^{10}q^2 + 520v^9 q^3 + 192v^8 q^4 + 48v^7 q^5 + 6v^6 q^6$$
$$+2196v^{11} + 4350v^{10}q + 5196v^9 q^2 + 4344v^8 q^3$$
$$+2628v^7 q^4 + 1114v^6 q^5 + 312v^5 q^6 + 52v^4 q^7 + 4v^3 q^8$$
$$+1620v^{10} + 4572v^9 q + 7413v^8 q^2 + 8284v^7 q^3 + 6732v^6 q^4$$
$$+4028v^5 q^5 + 1766v^4 q^6 + 556v^3 q^7 + 120v^2 q^8 + 16vq^9$$
$$+q^{10} \; . \tag{11.4}$$

(It can be checked that the $v = -1$ special case of this f coincides with the degeneracy per site, W, in Refs. [47].)

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